Comparing Inequality and Mobility in Economic Dynamics

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2014 Workshop on Wealth and Income Inequality in China and Singapore
at National University of Singapore, 5 June 2014

Introduction

- “Nothing struck me more forcefully than the general equality of conditions“, Tocqueville, *Democracy in America*
- Equality of Outcomes and Equality of Opportunity: Inequality of outcome can co-exist with equality of opportunity
- Equality of Outcome and Equality of Opportunity: when are they *complementary*; when are they *incompatible*?
- If *incompatible*, how to choose between the two
- Under what conditions are they *complementary*? (so that no need to choose between the two)

Introduction

- A common commitment: Liberty is to be guarded against the threats of coercion and oppression
- Disagreement about the nature of these threats
- In favour of *equality of opportunity*: excessive emphasis on equality of outcome could stifle human flourishing
- In favour of *equality of outcome*: inequality of outcome could lead to inequality of opportunity, to the detriment of talented individuals from less advantageous backgrounds
- Our contribution: providing a framework for this debate to be conducted

Modelling Framework

- Dynamic, Markovian
- Invariant distribution: exists, unique, globally stable for each Markov operator within a certain class
- Outcome (Inequality): characteristic of the invariant distribution,
- Opportunity (Mobility): dependence structure between two distributions after the invariant distributions has been reached

Inequality and Mobility

- Inequality: marginal distribution
- Mobility: how much the conditional distributions change with the initial conditions

Partial orderings available

- Inequality ranking: convex order, Lorenz order, second order stochastic dominance…
- Mobility ranking: copula dominance

Overview of Main Results

- Work with the class of Markov operator that generate unique and globally invariant distributions
- Thus, each operator F in this class is associated with a particular invariant distribution, and a particular dependence structure.
- Inequality: partial order on all invariant distributions
- Mobility: partial order on all possible dependence structures

Overview of Main Results

- Decomposition of

$$Q(t) \rightarrow Q(t+1)$$

into

$$Q(t) \rightarrow P(t) \rightarrow P(t+1) \rightarrow Q(t+1)$$

or,

$$Q(t) \rightarrow P(t) \rightarrow P(t+1) \rightarrow Q(t+1)$$

Overview of Main Results

$$Q(t) \rightarrow P(t) \rightarrow P(t+1) \rightarrow Q(t+1)$$

- Where $Q(t) \rightarrow P(t)$ and $P(t+1) \rightarrow Q(t+1)$ correspond to the *invariant distribution*
- and $P(t) \rightarrow P(t+1)$ correspond to the *dependence structure*

Overview of Main Results

$$Q(t) \to P(t) \to P(t+1) \to Q(t+1)$$

$$\overbrace{\textit{invariant distribution}}^{\text{inequality}} \to \overbrace{\textit{positional kernel}}^{\text{mobility}} \to \overbrace{\textit{invariant distribution}}^{\text{inequality}}$$

Overview of Main Results

- Copula s and positional kernels exactly correspond to each other
- positional kernels are completely characterized by their contours - referred to as "positional isoquants"
- Single crossing of every pair of positional isoquants implies copula dominance (mobility ranking)

Overview of Main Results

Naturally, two questions arise

1. Given a Markov operator, what is the invariant distribution, and equilibrium dependence structure? How do the invariant distribution and dependence structure change with the Markov operator?
2. Given any invariant distribution and dependence structure, it is possible to find the Markov operator that produces both? How does the Markov operator change with the invariant distribution and dependence structure?

Overview of Main Results

1. *Working backwards:* for any invariant distribution and any copula, it is (straightforward) to find a Markov operator that generates both (this means inequality and mobility are distinct concepts, any inequality-mobility combination is possible, and the operator that generate them can be found).

2. *Working forward*: under certain conditions, equality of opportunity is positively related with equality of outcome.

Literature

- Futia (1982), Hopenheyn and Prescott (1992) identify conditions that give rise to invariant distributions in Markovian enviroments, over compact sets.
- Starchurski (2008) extends the results to unbounded state spaces, and obtains global stability.
- Loury (1981) Intergenerational transfers and income distribution
- Literature on the measurement of mobility.

Inequality Ranking

Lorenz Order

- Lorenz curve: Basis for calculating the Gini coefficient.
- Lorenz curve dominance takes place when one Lorenz curve lies entirely above another.
- Unlike the Gini coefficient, Lorenz curve dominance, or the Lorenz order, is only a partial order.
- Lorenz order is closely related to the convex order.

Inequality Ranking

Mean Preserving Spread

- Rothschild and Stiglitz (1970)
- Implies that the c.d.f. of two distributions single cross.

Mobility Ranking

- Two dimensional copula: $C: [0,1]^2 \rightarrow [0,1]$ is a joint c.d.f. of two dimensional random vector with their marginal distributions being uniform on [0,1].

- Sklar's theorem: Given any two dimensional random vectors with marginal distributions joint distributions $\Lambda_1(.)$ and $\Lambda_2(.)$, the joint distribution can be written as

$$H(x_1, x_2) = C(\Lambda_1(x_1), \Lambda_2(x_2))$$

Mobility Ranking

- Copula is invariant under strictly increasing transforms (Nelson Theorem 2.4.3.)

- A partial order is Copula dominance. (Nelson Definition 2.8.1.) Copula C_1 is smaller than C_2 if $C_1(u, v) \leq C_2(u, v)$ for all u, v in $[0,1]^2$.

Model

- a given population, normalized to measure one
- Income of agent i at time t is given by $y_{i,t}$. The density and cumulative density of income distribution are denoted as $\phi_t(y)$ and $\Phi_t(y)$, respectively.

$$y_{i,t+1} = F\left(y_{i,t}, a_{i,t}\right),$$

- Ability $a_{i,t}$, i.i.d, with c.d.f $\Psi(.)$

Model

- A large amount of economic theories share this structure
- The $F(.,.)$ function itself is in a reduced form, in that the equilibrium actions of the agents are already endogenized.
- Changes in the economic environment, such as technological changes, policy changes can potentially change the shape of the $F(.,.)$ function.
- Economic growth can be accounted for by defining common growth factor A_t, such that $x_{i,t} \equiv y_{i,t}/A_t$,
- the common growth factor A_t can be a function of the agents' choices, however, it must impact all agents equally.

Conditions for Equilibrium

Three conditions needed for a globally stable invariant distribution to exist. (Starchurski, 2008; Theorem 8.2.14)

1. Positive probability of moving between any two points of the state space within finite time. This ensures that the whole distribution is ergodic.
2. The stochastic kernel satisfies geometric drift to the centre.
3. The stochastic kernel is bounded from above by a continuous function

Conditions for Equilibrium

Assumption 1. The slopes become very steep at the lowest end, and very flat at the higher end.

Assumption 2. $F(.,.)$ function is strictly increasing and continuous. $F(0,0) = 0$. (Note strictly increasing in crucial. Need to show that this ensures that the kernel is dominated by a continuous function.)

Conditions for Equilibrium

Proposition 1 Under Assumptions 1-2. there is a unique invariant distribution and it is globally stable.

- Let $\mathcal{M}_F$ denote the set of all F functions that satisfy Assumptions 1-2

Decomposition

- for any given cumulative distribution $\Phi(.)$, define rank as $\rho_t \equiv \Phi(y_t)$, such that $y_t = \Phi^{-1}(\rho_t)$.
- Then, for any given $\{F, \Phi\}$ pair, $y_{t+1} = F(y_t, a_t)$ can be rewritten as

$$\Phi^{-1}(\rho_{t+1}) = F(\Phi^{-1}(\rho_{t+1}), a_t),$$

Or,

$$\rho_{t+1} = \Phi[F(\Phi^{-1}(\rho_{t+1}), a_t)].$$

- ***Positional Kernel***: define $G(.,.): [0,1] \times [0,1] \rightarrow [0,1]$ as

$$G(\rho_t, \alpha_t) \equiv \Phi[F(\Phi^{-1}(\rho_t), \Psi^{-1}(\alpha_t))].$$

Decomposition

Definition For any $F(.,.)$ function, define the family of ***F-contours*** indexed by $m\epsilon\mathbb{R}^+$ as $\tau(a;m)$, which satisfies $F(\tau(a;m),a) = m$.

Each F-contour is a strictly decreasing function of a.

Decomposition

Definition For any positional kernel $G(.,.)$, define a family of ***positional isoquants*** indexed by $\rho \in [0,1]$ as $\pi(\alpha;\rho)$, which satisfies $G(\pi(\alpha;\rho), \alpha) = \rho$ for all $\rho \in [0,1]$.

- each positional isoquant is a strictly decreasing function of α
- the index ρ, which represents the "height" of the positional kernel along the particular positional isoquant, is pinned down by the area beneath the isoquant itself,

$$\int_0^1 \pi(\alpha;\rho)d\alpha = \rho$$

- each positional kernel is completely characterized by its family of positional isoquants.

Decomposition

Proposition 2 *Each positional kernel is uniquely represented by a copula.*

Proof: *Given* $\rho_{t+1} = G(\rho_t, \alpha_t)$*, it is possible to define*

$$c(z_1, z_2) = \iint \mathbb{I}[G(\rho_t, \alpha_t) \leq z_2]\mathbb{I}[\rho_t \leq z_1] dz_1 dz_2,$$

it is straightforward to verify that

$$c(z_1, z_2) = prob(\rho_{t+1} \leq z_2 | \rho_t \leq z_1)$$

is satisfied, therefore, $c(z_1, z_2)$ *is the copula that represents the positional kernel* $G(\rho_t, \alpha_t)$.

Decomposition

Definition: Let $\mathcal{C}$ denote the class of copula on the unit square $c(z_1, z_2)$ which satisfies $\partial c(z_1, z_2) / \partial z_1 > 0$, $\partial c(z_1, z_2) / \partial z_2 > 0$ and $\partial^2 c(z_1, z_2) / \partial z_1 \partial z_2 > 0$.

Proposition 3 *Each copula in $\mathcal{C}$ is uniquely represented by a positional kernel.*

Decomposition

Proof: *Given that each positional kernel is completely characterized by the family of positional isoquants it generates, it would be sufficient to show that each copula in* $\mathcal{C}$ *uniquely determines an entire family of positional isoquants. Now for any* $0 \leq z_2 \leq 1$, *denoting* $\pi^{-1}(z_1; z_2)$ *as the inverse function of* $\pi(.; z_2)$, *such that* $\pi(\pi^{-1}(z_1; z_2)\,; z_2) = z_1$; *the fact that* $\pi(.; z_2)$ *is monotone ensures that such an inverse function exists and is unique. Then the following relationship holds*

$$\frac{\partial c(z_1, z_2)}{\partial z_1} = \pi^{-1}(z_1; z_2),$$

and this gives the positional isoquant for every $0 \leq z_2 \leq 1$.

Decomposition

Theorem 4 *Under Assumptions 1-2, the dependence structure of every* $\{F, \Phi_F\}$ *pair is uniquely represented by a copula.*

Proof *The dependence structure of very* $\{F, \Phi_F\}$ *pair is uniquely represented by a positional kernel, and every positional kernel is uniquely represented by a copula* $c(z_1, z_2)$.

Working Backwards

Theorem 5 *For every distribution* ϕ *over* $\mathbb{R}^+$ *with strictly positive support, and for any copula* $c(.,.)$ *in* $\mathcal{C}$*, there is an unique operator represented by a* F *function such that* ϕ *is the invariant distribution, with the copula on* $\{\rho_t, \rho_{t+1}\}$ *being given by* $c(\rho_t, \rho_{t+1})$.

Working Backwards

Proof *Every copula* $c(.,.)$ *can be uniquely represented by a positional kernel* $G(\rho_t, \alpha_t)$*, which satisfies* $\rho_{t+1} = G(\rho_t, \alpha_t)$ *for all* $\{\rho_t, \rho_{t+1}\}$ *in the unit square and where* α_t *is uniformly distributed on the unit interval. Therefore, the* F *function that gives* ϕ *as its invariant distribution and* $c(\rho_t, \rho_{t+1})$ *as the copula is given by*

$$F(y_t, a_t) = \Phi^{-1}\left[G\left(\Phi(y_t), \Psi(a_t)\right)\right].$$

Working Backwards

Corollary 6 (Given any two operators and their corresponding invariant distributions and copulas, there is a third operator such that it generates the same invariant distribution as the first operator, and the same copula as the second operator.) *Given* $\{F, \Phi, c\}$ *and* $\{\hat{F}, \hat{\Phi}, \hat{c}\}$ *as two triplets of operators, invariant distributions, and copulas; there exists another triplet* $\{\tilde{F}, \Phi, \hat{c}\}$.

Working Backwards

Proof. The copula $\hat{c}$ can be represented by a positional kernel $\hat{G}$ which satisfies $\hat{G}(\rho_t, \alpha_t) \equiv \hat{\Phi}\left[\hat{F}\left(\hat{\Phi}^{-1}(\rho_t), \Psi^{-1}(\alpha_t)\right)\right]$.Thus, the operator $\tilde{F}$ that results is the same copula as $\hat{c}$ and the same invariant distribution as Φ is given by

$$\tilde{F}(y_t, a_t) = \Phi^{-1}\left[\hat{\Phi}\left[\hat{F}\left(\hat{\Phi}^{-1}\left(\Phi(y_t)\right), a_t\right)\right]\right].$$

Mobility Ordering and Positional Isoquants

Proposition 7 (Single crossing of all pairs of positional isoquants implies Copula dominance.) *Given two stochastic processes represented copula* $c(.,.)$ *and* $\hat{c}(.,.)$*, respectively; denoting their associated families of positional isoquants* $\{\pi(.;.)\}$ *and* $\{\hat{\pi}(.;.)\}$*, respectively; if for every* $0 \leq \rho \leq 1$*, there exists a* $0 \leq \alpha_0 \leq 1$ *such that, for all* $0 \leq \alpha \leq 1$*, the inequality* $(\alpha - \alpha_0)[\hat{\pi}(.;\rho) - \pi(.;\rho)] \geq 0$ *holds, then* $c(.,.)$ *dominates* $\hat{c}(.,.)$.

Mobility Ordering and Positional Isoquants

Proof *Let* $\{(\pi(.;\rho), \hat{\pi}(.;\rho))\}$ *be the pair of positional isoquants associated with some* $\rho \in [0,1]$. *By definition,* $\int_0^1 \pi(\alpha;\rho)d\alpha = \rho$ *for every* $0 \le \rho \le 1$ *(that is, the area underneath every positional isoquant is equal to the "height" it represents); therefore, the functions* $\pi(.;\rho)$ *and* $\hat{\pi}(.;\rho)$ *must intersects for some* $0 \le \alpha_0 \le 1$, *i.e.,* $\hat{\pi}(;\rho\alpha_0) - \pi(\alpha_0;\rho) = 0$. *Using the fact that* $\int_0^1 \pi(\alpha;\rho)d\alpha = \rho$, *and that* $c(z_1, z_2) = \int_0^1 min[\pi(\alpha; z_2), z_1]d\alpha$, *it must follow that* $c(z_1, z_2) \ge \hat{c}(z_1, z_2)$ *for any* z_1, z_2 *in the unit square.*

Mobility Inequality Link

Lemma 8 *For a given Markovian model* $\{F, \Phi, c\}$, *the families of* F*-contours* $\{\tau\}$ *and the family of positional isoquants* $\{\pi\}$; *the following holds for any given* $\alpha \in [0,1]$ *and* $z \in [0,1]$,

$$\pi\big(\Psi(a), \Phi(m)\big) = \Phi\big(\tau(a,m)\big)$$

where $\Phi(m) = z$ *and* $\Psi(a) = \alpha$.

Same Inequality, Different Mobility

Proposition 9 *For a given pair of Morkovian processes* $\{F, \Phi, c\}$ *and* $\{\hat{F}, \widehat{\Phi}, \hat{c}\}$, *if the invariant distributions are identical,* $\Phi = \widehat{\Phi}$*; but the copulas,* c *and* $\hat{c}$, *are different, in that* c *copula dominates* $\hat{c}$, *and that every pair of positional isoquants single cross, then it must follow that every pair of* F*-contour single cross.*

Same Mobility, Different Inequality

Proposition 10 *For a given pair of Morkovian processes* $\{F, \Phi, c\}$ *and* $\{\hat{F}, \hat{\Phi}, \hat{c}\}$, *if the copulas* c *and* $\hat{c}$ *are identical, then the F-contours must be related in the following way,*

$$\hat{\tau}\left(a, \hat{\Phi}^{-1}(z)\right) = \hat{\Phi}^{-1}\Phi\left(\tau\left(a, \Phi^{-1}(z)\right)\right)$$

for all $z \in [0,1]$ *and* $a \in \mathbb{R}^{+}$.

Mobility Inequality Link

A "neutral" experiment…

- Consider $\hat{F} = h(F)$, where $h(.)$ is a monotone increasing function.
- This ensures that the level sets are kept the same, thus, the change favours neither "ability" nor "bequest"
- Example: think of F as "performance", $h(.)$ as the mapping from performance to income
- Result: if unambiguous inequality and mobility ranking exists between the invariant distributions and the copulas corresponding to the two operators, then inequality and mobility are negatively related.

Mobility Inequality Link

"Single crossing" is the thread that connects inequality ranking (MPS iff c.d.f. single cross) with mobility ranking (Proposition 7). More formally…

Proposition 11 (equality of outcomes and equality of opportunity move in the same direction when the transformation of F function keeps the level sets unchanged.) *Let* $\hat{F} = h(F)$ *for some monotone increasing function h. Consider the given pair of Morkovian processes* $\{F, \Phi, c\}$ *and* $\{\hat{F}, \widehat{\Phi}, \hat{c}\}$*. If* $\widehat{\Phi}$ *is a mean preserving spread of* Φ *and that* c *and* $\hat{c}$ *can be strictly ordered, then* c *strictly copula dominates* $\widehat{c}$.

Mobility Inequality Link

Proof *Define* $\Phi(y_0) = \widehat{\Phi}(y_0) = z_0$ *to be where the two c.d.f. intersects. By Corollary 6, there must be a* $\tilde{F}$ *that generate the triplet* $\{\tilde{F}, \widehat{\Phi}, c\}$. *Since* $\hat{F} = h(F)$ *implies that both have the same family of F-contours. Thus, for any* $\hat{\tau}(a, y) = \tau(a, h^{-1}(y))$ *hold for any* (a, y). *Now since* $(y - y_0)\widehat{\Phi}^{-1}\Phi(y) \geq 0$, *by Proposition 10, the F-contours of* $\tilde{F}$ *and* F *single cross at* $\tilde{\tau} = \tau = y_0$. *Also, by Proposition 9, the F-contours of* $\tilde{F}$ *and* $\hat{F}$ *must single cross. Irrespective of whether* $\Phi^{-1}(z)$ *is greater or smaller than* $h^{-1}(\widehat{\Phi}^{-1}(z))$, *if there is a single cross transformation of the F-contours of* $\tilde{F}$ *such that* $\hat{\tau}(a, y) = \tau(a, h^{-1}(y))$ *holds, the direction must be in the direction of lowering mobility.*

Mobility Inequality Link

- **Intuition**: by making the mass of more spread out towards the tails, the ability-position tradeoff rate changes, making it harder to substitute for lower initial position by higher ability.

Summing Up

- A rigorous framework geared towards clarifying two concepts: equality of outcome and equality of opportunity
- Dynamic, Markovian environment; globally stable invariant distribution.
- Decomposition:

Invariant distribution ↔ Inequality ↔ Outcomes

Copula ↔ Positional Kernel ↔ Mobility ↔ Opportunity

- In general, any inequality-mobility combination is possible.
- There are environments where equalities of outcome and opportunity are complementary.